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Gauge covariant fermion propagator in quenched, chirally-symmetric quantum electrodynamics

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Abstract

We discuss the chirally symmetric solution of the massless, quenched, Dyson-Schwinger equation for the fermion propagator in three and four dimensions. The solutions are manifestly gauge covariant. We consider a gauge covariance constraint on the fermion–gauge-boson vertex, which motivates a vertex Ansatz that both satisfies the Ward identity when the fermion self-mass is zero and ensures gauge covariance of the fermion propagator.

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I. Introduction

The Dyson-Schwinger equations (DSEs) are a valuable nonperturbative tool for studying field theories; recent reviews of this approach can be found in Refs. [1–3]. While this approach has limitations, which are being addressed, it greatly facilitates the development of models which bridge the gap between short-distance, perturbative QCD and the extensive amount of low- and intermediate-energy phenomenology in a single, covariant framework. In addition, as the approach continues to be developed, there are obvious points where cross-fertilisation with lattice studies will become valuable. One such example is the study of the phenomenological implications of gluon propagators obtained in lattice simulations [4].

In recent years a good deal of progress has been made in addressing the limitations of the DSE approach in the study of Abelian gauge theories. Calculations are now such that direct comparison can be made between quantities calculated in the DSE approach and those calculated in lattice simulations. One illustrative example is the gauge invariant fermion condensate, $\langle\bar{\psi}\psi\rangle$. This has been calculated in three-dimensional QED and the agreement with reported lattice results [5] is very good [3,6]. This progress has been made possible by a realisation of the importance of the fermion–gauge-boson vertex.

The fermion–gauge-boson vertex satisfies a DSE. This equation involves the kernel of the fermion-antifermion Bethe-Salpeter equation, which cannot be expressed in a closed form; i.e., its skeleton expansion has infinitely many terms. Given this, research has concentrated on placing physically reasonable constraints on the form of the vertex, constructing simple Ansätze that embody them, employing a given Ansatz in the DSE for the fermion propagator, and studying the properties of the solution. This has lead to an understanding of the role of the vertex in ensuring multiplicative renormalisability [7] and gauge covariance [8], within the class of linear, covariant gauges.

Recently a gauge covariance constraint on a class of vertices, applicable to chirally-symmetric, quenched QED, was proposed [8]. One characteristic of this class of vertices is that the DSE admits the free propagator solution in Landau gauge. This constraint was used in a critical analysis of three often used vertex Ansätze. Of these three, that of Ref. [7] was the only one not eliminated. [Herein chirally symmetric means that that the fermion bare mass is zero and there is no dynamical mass generation and quenched means that the vacuum polarisation is neglected.] This Ansatz has been used in studies of three- and four-dimensional QED [8] and in phenomenological studies of QCD [4]. The fermion-DSE in four-dimensional, chirally symmetric, quenched QED was first studied using this Ansatz in Ref. [9], however, the explicit form of the solution obtained therein is not gauge covariant, inconsistent with the expectations of Ref. [8].

Herein we are interested in analysing, and making explicit, the restrictions on the fermion–gauge-boson vertex imposed by the gauge covariance constraint proposed in Ref. [8]. To this end, we solve the fermion-DSE in three- and four-dimensional QED and obtain manifestly gauge covariant solutions in both cases. In the process we show how the constraint is crucial to this outcome and identify an error in Ref. [9] that leads to the disagreement with Ref. [8]: it arises because of an inappropriate regularisation procedure. We show that, although the gauge-covariance constraint is satisfied by the chirally-symmetric limit of the Ansatz of Ref. [7], this vertex violates the Ward identity and is hence unsuitable in this application. We use the constraint to construct an Ansatz that overcomes this defect. Our

study shows that the gauge-covariance constraint allows a large class of vertex Ansätze, of which those proposed so far are simple examples.

II. Solution of the properly regularised fermion-DSE

In Euclidean metric, with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $\gamma_\mu^\dagger = \gamma_\mu$, the unrenormalised fermion-DSE in quenched, massless, d=3- or 4-dimensional QED is

$$S^{-1}(p) = i\gamma \cdot p + e^2 \int \frac{d^{\rm d}q}{(2\pi)^{\rm d}} D_{\mu\nu}(p-q)\, \gamma_\mu\, S(q)\, \Gamma_\nu(q,p). \tag{1}$$

We use a 4×4 representation of the Euclidean Dirac algebra in 3 and 4 dimensions, which allows for a study of dynamical chiral symmetry breaking without parity violation [10]. For d=3 one can use, for example, γ_1, γ_2, γ_4 with $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$, where the matrices have their usual Euclidean definitions. Using the covariant gauge fixing procedure, as we do herein, the fermion propagator in Eq. (1) has the general form $S^{-1}(p) = i\gamma \cdot p\, A(p^2) + B(p^2)$.

The quenched photon propagator in Eq. (1) is

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} - k_\mu k_\nu \hat{\Delta}(k;\xi) \equiv D^{\rm T}_{\mu\nu}(k) - k_\mu k_\nu \hat{\Delta}(k;\xi) \tag{2}$$

where $\hat{\Delta}(k;\xi)$ is the gauge fixing term, with ξ the gauge parameter. One particular choice is

$$\hat{\Delta}(k;\xi) = -\xi\, \frac{1}{(k^2)^2}\, R\left(\frac{k^2}{\Lambda^2}\right) \;, \tag{3}$$

where Λ^2 is a regularising parameter and

$$R(x) > 0 \;\forall x \;, \quad \text{with} \;\; R(0) = 1 \quad \text{and} \quad \int_0^\infty dx\, R(x) = 1 \;. \tag{4}$$

Equation (3) reduces to the standard covariant gauge fixing term when $\Lambda^2 \to \infty$.

Defining the function

$$\Delta(x;\xi) = \int \frac{d^{\rm d}k}{(2\pi)^{\rm d}}\, \hat{\Delta}(k;\xi)\, {\rm e}^{ik\cdot x} \tag{5}$$

then the fermion propagator in the gauge specified by ξ, $S(x,\Delta)$, is obtained from that in Landau gauge, $\xi = 0$; i.e., $S(x,0)$, via the Landau-Khalatnikov-Fradkin [11–13] (LKF) transformation

$$S(x;\Delta) = S(x,0){\rm e}^{e^2[\Delta(0;\xi) - \Delta(x;\xi)]} \;. \tag{6}$$

The fermion–gauge-boson vertex in Eq. (1) can be written in the general form

$$\Gamma_\mu(p,q) = \Gamma^{\rm BC}_\mu(p,q) + \sum_{i=1}^{8} T^i_\mu(p,q)\, g^i(p^2, p\cdot q, q^2) \tag{7}$$

where [14]

$$\Gamma^{\rm BC}_\mu(p,q) = \Sigma_A(p,q)\,\gamma_\mu + (p+q)_\mu \left\{ \Delta_A(p,q)\,\tfrac{1}{2}\,[\gamma\cdot p + \gamma\cdot q] - i\Delta_B(p,q)\right\} \;, \tag{8}$$

with

$$\Sigma_A(p,q) \equiv \tfrac{1}{2}\,[A(p^2)+A(q^2)] \quad \text{and} \quad \Delta_A(p,q) \equiv \frac{A(p^2)-A(q^2)}{p^2-q^2} \;, \tag{9}$$

and similarly for Δ_B.

In Eq. (7), $T^i_\mu(p,q)$ are eight tensors, transverse with respect to $(p-q)_\mu$, given in Eq. (A1) of the Appendix. Under charge conjugation the vertex transforms as follows [15]

$$[\Gamma_\mu(-q,-p)]^{\rm T} = -\mathcal{C}\,\Gamma_\mu(p,q)\,\mathcal{C}^\dagger \;, \tag{10}$$

where "T" denotes matrix transpose and $\mathcal{C}=\gamma_2\gamma_4$ is the charge conjugation matrix, which entails that all of the functions g^i are symmetric under $p \leftrightarrow q$ except for g^6, which is antisymmetric.

In considering Eq. (1) alone the functions g^i are undetermined although, in principle, they can be calculated within the DSE framework. As we have remarked, however, this is a difficult and unsolved problem. Hitherto, progress toward determining these functions has been made by studying the implications of the following constraints, which the fermion–gauge-boson vertex in Abelian gauge theories must satisfy [8]: the vertex must A) satisfy the Ward-Takahashi identity; B) be free of kinematic singularities [i.e., have a well defined limit as $p\to q$]; C) reduce to the bare vertex in the free field limit in the manner prescribed by perturbation theory; D) transform under charge conjugation as indicated in Eq. (10) and preserve the Lorentz symmetries of the theory; E) ensure local gauge covariance of the propagators; and F) ensure multiplicative renormalisability of the DSE in which it appears.

In general, Eq (1) admits both chirally asymmetric, $B\neq 0$, and chirally symmetric, $B=0$, solutions. Herein we focus on the latter, in which case the propagator has the form

$$S(p) = \frac{1}{i\gamma\cdot p\,A(p^2)} = \frac{\mathcal{F}(p^2)}{i\gamma\cdot p} \;. \tag{11}$$

Only the tensors $i=2,3,6,8$ in Eq. (7) contribute in this case and the active part of the vertex can therefore be written as

$$\Gamma_\mu(p,q) = \Gamma^{\rm BC}_\mu(p,q)\Big|_{B=0} + \Delta_A(p,q) \sum_{i=2,3,6,8} T^i_\mu(p,q)\, f^i(p^2,p\cdot q,q^2) \;. \tag{12}$$

Substituting Eqs. (2), (11) and (12) into Eq. (1) yields

$$S^{-1}(p) = i\gamma\cdot p - \left\{ e^2 \int \frac{d^{\rm d}q}{(2\pi)^{\rm d}}\, i\gamma\cdot(p-q)\,\hat\Delta(p-q;\xi)\right\} \tag{13}$$

$$+e^2 \int \frac{d^{\rm d}q}{(2\pi)^{\rm d}}\, i\gamma\cdot(p-q)\hat\Delta(p-q;\xi) S(q)\, S^{-1}(p) + e^2 \int \frac{d^{\rm d}q}{(2\pi)^{\rm d}} D^{\rm T}_{\mu\nu}(p-q)\gamma_\mu S(q)\Gamma_\nu(q,p) \;.$$

The parenthesised term in Eq. (13) is zero in any translationally invariant regularisation scheme. In Ref. [9] a hard cutoff was used, which violates translational invariance, leading to

a spurious additional term in the massless, chirally symmetric DSE. This is why the solution obtained therein is not gauge covariant.

In the quenched approximation and in the absence of dynamical chiral symmetry breaking, a sufficient condition for gauge covariance of the fermion propagator is that

$$\int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}} D^{\mathrm{T}}_{\mu\nu}(p-q)\gamma_\mu S(q)\Gamma_\nu(q,p) = 0 \;, \tag{14}$$

for arbitrary $\hat{\Delta}(k;\xi)$ [8]. This leads to constraints on, and relations between, the functions f^i, which we discuss in Sec. III. With a vertex satisfying Eq. (14), Eq. (13) reduces to the following linear equation for $\mathcal{F}(p^2)$:

$$1 = \mathcal{F}(p^2) + e^2 \int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}}\,(p-q)\cdot q\,\hat{\Delta}(p-q;\xi)\frac{\mathcal{F}(q^2)}{q^2} \;. \tag{15}$$

In the following we discuss the cases d=3 and 4.

II.1. Three-dimensional QED

The case d=3 was considered explicitly in Ref. [8]. In this case the regularising parameter can be removed, $\Lambda^2 \to \infty$, and the equation takes the form

$$\mathcal{F}(p) = -\frac{\alpha\xi}{2p}\int_0^\infty dq\,q\,\mathcal{F}(q)\,\frac{d}{dq}\left(\frac{1}{q}\ln\left|\frac{p+q}{p-q}\right|\right) \;, \tag{16}$$

where $\alpha = e^2/(4\pi)$, and the solution is

$$\mathcal{F}(p) = 1 - \frac{\alpha\xi}{2p}\arctan\left(\frac{2p}{\alpha\xi}\right) \;. \tag{17}$$

This is just the LKF transform of the free fermion propagator, which is as it must be since that is the solution in Landau gauge.

II.2. Four-dimensional QED

In studying d=4 we consider the renormalised form of this equation:

$$1 = \mathcal{Z}_2\mathcal{F}_R(p) + \mathcal{Z}_2 e^2 \int \frac{d^4q}{(2\pi)^4}\,(p-q).q\,\hat{\Delta}(p-q;\xi)\frac{\mathcal{F}_R(q)}{q^2} \;, \tag{18}$$

where $\mathcal{Z}_2\mathcal{F}_R(p) = \mathcal{F}(p)$, which, because of multiplicative renormalisability, is expected to have a power law solution [16]: $\mathcal{F}_R = (p^2/\mu^2)^{\phi(\xi)}$, where $\phi(\xi)$ is not determined by the constraint of multiplicative renormalisability. In Landau gauge $\hat{\Delta}(x;\xi=0) = 0$ and hence from Eq. (18) $\mathcal{Z}_2^{\xi=0}\mathcal{F}_R(p) = 1$; i.e., $\phi(\xi=0) = 0$. Renormalising such that one has the free, massless fermion propagator as the solution in Landau gauge then

$$\mathcal{Z}_2^{\xi=0} = 1 \;. \tag{19}$$

It follows from Eq. (18) that

$$\mathcal{F}_R(p_2) - \mathcal{F}_R(p_1) = e^2 \int \frac{d^4q}{(2\pi)^{\rm d}} \frac{\mathcal{F}_R(q)}{q^2} \Big((p_1 - q).q\, \hat{\Delta}(p_1 - q; \xi) - (p_2 - q).q\, \hat{\Delta}(p_2 - q; \xi)\Big) \ . \tag{20}$$

After evaluating the angular integral, the right-hand-side of Eq. (20) is finite when the regularising parameter is removed, $\Lambda^2 \to \infty$, and this yields

$$\mathcal{F}_R(p^2) - \mathcal{F}_R(\mu^2) = -\frac{\alpha\xi}{4\pi} \int_{p^2}^{\mu^2} dq^2\, \frac{\mathcal{F}_R(q)}{q^2} \ . \tag{21}$$

The solution of this equation is

$$\mathcal{F}_R(p^2) = \mathcal{F}_R(\mu^2) \left(\frac{p^2}{\mu^2}\right)^{\frac{\alpha\xi}{4\pi}} \ , \tag{22}$$

which is multiplicatively renormalisable and gauge covariant (again, this is the LKF transform of the free fermion propagator, as it must be).

The renormalisation constant in an arbitrary gauge follows from Eq. (6) [12]

$$\mathcal{Z}_2^{\xi} = \mathcal{Z}_2^{\xi=0}\, {\rm e}^{e^2\Delta(0;\xi)} \ . \tag{23}$$

Defining $H(x) = e^2[\Delta(0;\xi) - \Delta(x;\xi)]$, one finds from Eqs. (3) and (5) that

$$x^2\, H'(x^2) = -\nu \int_0^{\infty} dk\, \frac{2}{k}\, J_2(kx)\, R\left(\frac{k^2}{\Lambda^2}\right) \tag{24}$$

where $\nu = (\alpha\xi)/(4\pi)$ and J_2 is a Bessel function.

For Λ sufficiently large the properties of $R(x)$ given in Eq. (4) entail that

$$x^2\, H'(x^2) \approx -\nu \int_0^{\infty} dk\, \frac{2}{k}\, J_2(kx)\, \exp\left(-\frac{k^2}{\hat{\Lambda}^2}\right) \ , \tag{25}$$

from which it follows that

$$x^2\, H'(x^2) \approx -\nu \frac{\hat{\Lambda}^2 x^2}{1 + \hat{\Lambda}^2 x^2} \ , \tag{26}$$

for some $\hat{\Lambda} \propto \Lambda$. Hence, introducing the scale μ^2,

$$e^2\Delta(x;\xi) = \nu\, \ln\left(\frac{\mu^2}{\Lambda^2} + \mu^2 x^2\right) \ . \tag{27}$$

Using Eq. (19), this gives

$$\mathcal{Z}_2^{\xi} = \left(\frac{\mu^2}{\Lambda^2}\right)^{\frac{\alpha\xi}{4\pi}} \ . \tag{28}$$

III. Gauge covariance and vertex Ansätze

The question as to the form of the vertex that satisfies Eq. (14) arises. Substituting Eq. (12) into Eq. (14) leads to

$$
\begin{aligned}
0 = \int \frac{d^{\mathrm{d}}q}{(2\pi)^{\mathrm{d}}} \frac{1}{(p-q)^2} \frac{A(p^2)-A(q^2)}{p^2-q^2} \times \qquad (29) \\
\Bigg(p^2q^2 + \tfrac{1}{2}(p^2+q^2)p\cdot q - \tfrac{1}{2}\, p\cdot q\, \frac{(p^2-q^2)^2}{(p-q)^2} - f^2(p^2+q^2)[p^2q^2-(p\cdot q)^2] \\
+2f^3[(p\cdot q)^2 + p^2q^2 - p\cdot q\,(p^2+q^2)] + f^6(d-1)p\cdot q(p^2-q^2) + f^8(2-\mathrm{d})[p^2q^2-(p\cdot q)^2] \Bigg) \; .
\end{aligned}
$$

From this it is clear that the vertex, $\Gamma_\mu(p,q)$, which ensures gauge covariance and multiplicative renormalisability will, in general, depend on p^2, q^2, $p{\cdot}q$ and the ratio $\rho(p,q) = A(p)/A(q)$. However, there are simple, ρ-independent choices for the functions f^i for which the vertex satisfies Eq. (29).

To illustrate this we assume that the functions f^i in Eq. (12) are independent of $p\cdot q$. In this case one can evaluate the angular integrals in Eq. (29) to find

$$
\begin{aligned}
0 = \int_0^\infty dq^{\mathrm{d}-2}\, \mathcal{F}(q^2)\, \Delta_A(p,q) \, \Bigg(\tfrac{1}{2}\,(\mathrm{d}-1)I_3 + (\mathrm{d}-1)\frac{p^2-q^2}{p^2+q^2} I_3 f^6(q^2,p^2) \qquad (30) \\
+ [I_1 - I_3] \left[f^3(q^2,p^2) - \tfrac{1}{2}\left[p^2+q^2\right]^2 f^2(q^2,p^2) + \left(1-\tfrac{\mathrm{d}}{2}\right) f^8(q^2,p^2) \right] \Bigg) \; ,
\end{aligned}
$$

where the $I_i(p^2,q^2)$ are given in Eq. (A2) and we have used Eqs. (A7) and (A8).

III.1. Curtis-Pennington Ansatz

It is clear that the choice

$$
f^6(p^2,q^2) = \tfrac{1}{2}\,\frac{p^2+q^2}{p^2-q^2} \quad \text{with} \quad f^i = 0,\; i \neq 6 \qquad (31)
$$

ensures that Eq. (30) is satisfied. This is just the chirally symmetric limit of the vertex proposed in Ref. [7] and so we see that this Ansatz leads to a solution of the chirally symmetric, quenched fermion-DSE that is gauge covariant and multiplicatively renormalisable, as discussed in Ref. [8]. [Equation (35) in Ref. [8] is the equation given in Ref. [9]. As we remarked above, the fact that the parenthesised term in Eq. (13) is zero eliminates the first term on the right-hand-side of this equation, which becomes Eq. (15) above.]

There is a problem with this vertex, however:

$$
\lim_{p\to q} f^6(p^2,q^2)\, T^6_\mu(p,q) = \text{indeterminate} \; . \qquad (32)
$$

Therefore, in the chirally symmetric case, $B=0$, this vertex violates criterion B) and hence does not satisfy the Ward identity.

III.2. A Ward identity preserving Ansatz

A simple Ansatz satisfying Eq. (14) and all of the criteria listed in Sec. II, and hence one that preserves the Ward identity in the chiral limit, is $f^i = 0$ for $i \neq 3, 8$ with

$$f^3 = \tfrac{1}{2}\left(\tfrac{\mathrm{d}}{2} - 1\right) f^8 \quad \text{and} \quad f^8 = \frac{1}{\frac{\mathrm{d}}{2} - 1} \frac{(\mathrm{d}-1) I_3}{I_1 - I_3} \ . \tag{33}$$

For d=3, I_1 has a logarithmic divergence as $p \to q$ [see Eq. (A4)] but nevertheless

$$\lim_{p \to q} f^8(p^2, q^2)\, T^8_\mu(p, q) = 0 \ , \tag{34}$$

with a similar result for the T^3_μ term. Hence, criterion B) is satisfied and the vertex preserves the Ward identity.

For d=4 the explicit form of this vertex Ansatz is

$$f^3 = \tfrac{1}{2} f^8 \quad \text{with} \quad f^8 = \begin{cases} \frac{3\left(p^2+q^2\right)}{3p^2 - q^2} \ , & p^2 > q^2 \\ \frac{3\left(p^2+q^2\right)}{3q^2 - p^2} \ , & p^2 < q^2 \end{cases} \ . \tag{35}$$

It will be observed that for $p^2 >> q^2$ one recovers the O(α)-corrected perturbative vertex in the leading logarithm approximation with this Ansatz [7], consistent with constraint C) above. It is the simplest Ansatz which both does this and preserves the Ward identity in the chirally symmetric limit, $B = 0$.

III.3. A more general, Ward identity preserving Ansatz

For $d = 4$, a more general Ansatz can be obtained by writing Eq. (30) in the following form:

$$0 = \int_0^1 dx \left(x[1 - \rho(x)] - \frac{1}{x}\left[1 - \frac{1}{\rho(x)}\right]\right) \left\{ 3 f^6(1,x) + \tfrac{3}{2}\,\frac{1+x}{1-x} + h(1,x)\,\frac{3-x}{1-x} \right\} \tag{36}$$

with $\rho(x) \equiv A(p^2)/A(q^2)$ and

$$h(1,x) = f^3(1,x) - \tfrac{1}{2}(1+x) f^2(1,x) - f^8(1,x) \ . \tag{37}$$

In deriving Eq. (36) we used the symmetry properties: $f^i(1,x) = f^i(1,x^{-1})$ for $i \neq 6$ and $f^6(1,x) = -f^6(1,x^{-1})$ and the explicit forms of the angular integrals given in the appendix.

Taking into account the symmetry properties of f^i and criterion B) one can introduce a function $F(x)$ with the properties

$$\text{a)} \ \int_0^1 dx\, F(x) = 0 \quad \text{and} \quad \text{b)} \ F(1) + F'(1) = 6\rho'(1) \tag{38}$$

in terms of which Eq. (36) is solved by

$$h(1,x) = \tfrac{1}{4}\frac{F(x) - F(1/x)}{x[1-\rho(x)] - \dfrac{1}{x}\left[1 - \dfrac{1}{\rho(x)}\right]}\ , \tag{39}$$

$$f^6(1,x) = -\left[\tfrac{1}{2} + \tfrac{1}{3}h(1,x)\right]\frac{1+x}{1-x} + \tfrac{1}{6}\frac{F(x) + F(1/x)}{x[1-\rho(x)] - \dfrac{1}{x}\left[1 - \dfrac{1}{\rho(x)}\right]}\ . \tag{40}$$

Condition b) in Eq. (38) ensures that $f^6(1,1)$ is finite and hence that the vertex is free of kinematic singularities.

In order to further constrain $F(x)$, we note that since the denominator in Eq. (39) is of $\mathrm{O}(\alpha)$ in the perturbative expansion of $\rho(x)$ for small ν, one may also require that $F(x)$ be of $\mathrm{O}(\alpha)$ in this case. One example of a model that satisfies this constraint and Eq. (38) is

$$F(x) = 6\rho'(1)\,\frac{1 - \omega\, x^{\omega-1}}{1-\omega^2}\ , \tag{41}$$

with $\omega > 0$ and $\mathrm{O}(\alpha)$.

Equations (12), (37) and (39-41) provide a $\rho(x)$ dependent, Ward identity preserving vertex Ansatz.

IV. Summary and conclusions

We have studied the Dyson-Schwinger equation (DSE) for the fermion propagator in quenched, massless three- and four-dimensional QED obtaining the chirally symmetric solution in both cases. The solutions are gauge covariant and multiplicatively renormalisable.

The solution obtained in a previous study [9] of the four-dimensional case is incorrect. We showed that the error arises because of an inappropriate regularisation of the DSE.

We employed a constraint equation, proposed in Ref. [8], to demonstrate that gauge covariance of the solution restricts the form of the fermion–gauge-boson vertex and showed that, in general, the vertex can depend on the ratio $\rho(p,q) = A(p^2)/A(q^2)$. It is not difficult to construct explicit examples of this type, however, simpler forms are possible. The Ansatz proposed in Ref. [7] is one such form, however, with $B = 0$, it violates the Ward identity and is therefore not suitable for studies of the chirally symmetric fermion-DSE. We proposed a minimal, alternate form which overcomes this defect. Other forms are possible and all can be constructed using the constraint equation.

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APPENDIX:

Vertex tensors.

The eight tensors in Eq. (7) are, with $k = p - q$,

$$\begin{aligned}
&T^1_\mu = -ip_\mu(k\cdot q) + iq_\mu(k\cdot p)\ ;\ T^2_\mu = -i\gamma\cdot(p+q)T^1_\mu\ ;\\
&T^3_\mu = \gamma_\mu k^2 - k_\mu\gamma\cdot k\ ;\qquad\qquad T^4_\mu = -\tfrac{1}{2}[\gamma\cdot p,\gamma\cdot q]T^1_\mu\ ;\\
&T^5_\mu = \tfrac{1}{2}[\gamma_\mu, i\gamma\cdot k]\ ;\ T^6_\mu = \gamma_\mu\,(p^2-q^2) - (p+q)_\mu\,\gamma\cdot(p-q)\ ;\\
&T^7_\mu = -\tfrac{i}{2}\,(p^2-q^2)\,[\gamma_\mu\,\gamma\cdot(p+q) - (p+q)_\mu] - \tfrac{i}{2}\,(p+q)_\mu[\gamma\cdot p,\gamma\cdot q]\ ;\\
&T^8_\mu = \tfrac{1}{2}\,[\gamma\cdot p\,\gamma\cdot q\,\gamma_\mu - \gamma_\mu\,\gamma\cdot q\,\gamma\cdot p]\ .
\end{aligned}\tag{A1}$$

Useful integrals.

The integrals in Eq. (30) are

$$\begin{aligned}
&I_1 = p^2q^2\mathcal{I}_1\ ;\qquad\quad I_2 = \tfrac{1}{2}\left([p^2+q^2]\,\mathcal{I}_1 - 1\right)\ ;\\
&I_3 = \tfrac{1}{2}\,[p^2+q^2]\,I_2\ ;\ I_4 = \tfrac{1}{4}\,[p^2-q^2]^2\left([p^2+q^2]\,\mathcal{I}_2 - \mathcal{I}_1\right)
\end{aligned}\tag{A2}$$

where

$$\mathcal{I}_n = \int d\Omega_{\rm d}\,\frac{1}{(p-q)^{2n}}\tag{A3}$$

with $\int d\Omega_{\rm d} \equiv [\frac{1}{\mathcal{N}}\int_0^\pi d\theta_2\,\sin^{{\rm d}-2}\theta_2\int_0^\pi d\theta_3\,\sin^{{\rm d}-3}\theta_3\ldots\int_0^{2\pi} d\theta_{{\rm d}-1}]$ and $\mathcal{N} = 2\pi^{{\rm d}/2}/\Gamma({\rm d}/2)$. For d=3

$$\mathcal{I}_1 = \frac{1}{2pq}\ln\left(\frac{(p+q)^2}{(p-q)^2}\right)\quad\text{and}\quad \mathcal{I}_2 = \frac{2}{(p^2-q^2)^2}\tag{A4}$$

while for d=4

$$\mathcal{I}_1 = \frac{1}{p^2}\theta(p^2-q^2) + \frac{1}{q^2}\theta(q^2-p^2)\quad\text{and}\quad \mathcal{I}_2 = \frac{1}{|p^2-q^2|}\,\mathcal{I}_1\ .\tag{A5}$$

It can be shown that, for arbitrary d,

$$({\rm d}-3)\,\left(p^2+q^2\right)\,\mathcal{I}_1 + \left(p^2-q^2\right)^2\,\mathcal{I}_2 = {\rm d}-2\tag{A6}$$

from which many useful relations follow; for example,

$$0 = \int d\Omega_{\rm d}\,\frac{1}{(p-q)^2}\,\left(({\rm d}-3)\,p\cdot q + 2\,\frac{p\cdot(p-q)\,(p-q)\cdot q}{(p-q)^2}\right)\ ,\tag{A7}$$

$$0 = I_1 + ({\rm d}-2)I_3 - I_4\ .\tag{A8}$$

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